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Automorphisms and regular embeddings of merged Johnson graphs

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Abstract

The merged Johnson graph $J(n,m)_I$ is the union of the distance i graphs $J(n,m)_i$ of the Johnson graph $J(n,m)$ for $i \in I$, where $\emptyset \neq I \subseteq \{1,\ldots,m\}$ and $2 \leq m \leq n/2$. We find the automorphism groups of these graphs, and deduce that their only regular embedding in an orientable surface is the octahedral map on the sphere for $J(4,2)_1$, and that they have just six non-orientable regular embeddings. This yields classifications of the regular embeddings of the line graphs $L(K_n) = J(n,2)_1$ of complete graphs, their complements $\overline{L(K_n)} = J(n,2)_2$, and the odd graphs $O_{m+1} = J(2m+1,m)_m$.

1. Introduction

A standard problem in topological graph theory is that of determining the regular embeddings, in orientable or non-orientable surfaces, of a given class of arc-transitive connected graphs. Here we use the term ‘regular’ in the sense of Coxeter and Moser [7, Chapter 8], meaning that the map has two specific automorphisms, cyclically permuting the successive edges around a face and around a vertex meeting that face. This problem has been solved for several classes of graphs: for instance, the work of Biggs [2] and of James and the author [19] yields a classification of the orientable regular embeddings of complete graphs, while their non-orientable regular embeddings are described by James in [18]. Nedela and Škoviera [26] have obtained similar results for other classes of graphs, such as cocktail party graphs and dipoles.

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Here we consider a class of graphs based on the Johnson graphs. The vertices of a Johnson graph $J(n, m)$ are the m-element subsets of an n-element set $(2 \leq m \leq n/2)$, adjacent if their intersection has $m-1$ elements. Given a nonempty subset $I \subseteq \{1, \ldots, m\}$, we define the merged Johnson graph $J(n, m)_I$ to be the union of the distance i graphs $J(n, m)_i$ of $J(n, m)$ for all $i \in I$, so two m-element subsets are adjacent in $J(n, m)_I$ if their intersection has $m - i$ elements for some $i \in I$. The graphs $J(n, m)_I$ include many interesting graphs, such as the Johnson graphs $J(n, m) = J(n, m)_1$, the line graphs $L(K_n) = J(n, 2)_1$ of the complete graphs, their complements $J(n, 2)_2$, and the odd graphs $O_{m+1} = J(2m + 1, m)_m$.

In Theorem 2 we determine the automorphism group of each graph $J = J(n, m)_I$, using the work of Ustimenko-Bakumovskiĭ [29, 30] and others on overgroups of the symmetric group S_n in its action on m-element subsets: by considering the orbitals of each overgroup, we determine the sets I for which it leaves $J(n, m)_I$ invariant. We then use results of Livingstone and Wagner [25] and Kantor [23] on m-homogeneous permutation groups, and of Zassenhaus [35] and Gorenstein and Hughes [15] on triply transitive permutation groups, to determine which subgroups of $\mathrm{Aut}\, J$ can act as automorphism groups of regular embeddings of J. We show in Theorem 10 that there are, up to isomorphism, just seven such embeddings: these are the well-known octahedral embedding of $J(4, 2)_1 = J(4, 2) = L(K_4)$ in the sphere, and six regular embeddings of $J(4, 2)_1$, $J(4, 2)_{1,2} = K_6$, $J(5, 2)_1 = L(K_5)$ and Petersen's graph $J(5, 2)_2 = \overline{L(K_5)}$ in non-orientable surfaces; constructions of some of these maps are considered in Section 8. In fact, not only do almost all of these graphs J have no regular embeddings, we show in Theorem 14 that most of them have no vertex-transitive embeddings, though in this case we have not attempted to find a complete classification.

Although this paper uses some quite deep results on finite permutation groups, they are independent of the classification of finite simple groups, so our methods could be regarded as 'elementary' in some sense. Nevertheless, it would be interesting to investigate a more combinatorial approach, perhaps replacing permutation groups with association schemes. For general background on permutation groups, see the book by Dixon and Mortimer [8].

2. Johnson graphs and their mergings

Let N be a finite set of n elements, and let Ω denote the set $N^{\{m\}}$ of m-element subsets $M \subseteq N$, for some fixed m such that $1 \leq m \leq n - 1$, so $d := |\Omega| = \binom{n}{m}$. The *Johnson graph* $J(n, m)$ has vertex-set Ω, with vertices M and M' joined by an edge if and only if $|M \cap M'| = m - 1$ (see [3, Section 9.1] for a detailed study of the properties of this graph). The distance between two vertices M and M' of $J(n, m)$ is $|M \setminus M'| = |M' \setminus M|$, so $J(n, m)$ has diameter $\min\{m, n - m\}$. The *distance i Johnson graph* $J(n, m)_i$, where $0 \leq i \leq \min\{m, n - m\}$, also has vertex-set Ω, with M and M' joined by an edge if and only if they are at distance i in $J(n, m)$, that is, $|M \cap M'| = m - i$; in particular, $J(n, m)_1 = J(n, m)$. For fixed n and m, the graphs $J(n, m)_i$ are the orbital graphs for the action of S_n on Ω, induced by its natural action on N: each $J(n, m)_i$ corresponds to the orbital (or 2-orbit, that is, orbit of S_n on Ω^2)

$$\Gamma_i = \{(M, M') \in \Omega^2 \mid |M \cap M'| = m - i\}$$

(see [8, Section 3.2] and [27] for orbital graphs). We denote by $\Gamma_i(M)$ the set of neighbours of M in $J(n,m)_i$: these are the subsets $M' = (M \setminus I_1) \cup I_2$ of N, where I_1 and I_2 are i-element subsets of M and of $\overline{M} = N \setminus M$, so $J(n,m)_i$ has valency

$$|\Gamma_i(M)| = \binom{m}{i}\binom{n-m}{i}.$$

Included among the graphs $J(n,m)_i$ are the complete graphs $K_n = J(n,1)_1$, the null (or empty) graphs $\overline{K}_n = J(n,1)_0$, the line graphs $L(K_n) = J(n,2)_1$ of the complete graphs, and their complements $\overline{L(K_n)} = J(n,2)_2$. In particular, $J(5,2)_2$ is Petersen's graph, and more generally, $J(n,m)_m$ is the Kneser graph $K(n,m)$ [3, p. 258], with $J(2m+1,m)_m = O_{m+1}$, the odd graph of valency $m+1$ [3, p. 259].

Complementation of subsets $M \mapsto \overline{M}$ induces an isomorphism $J(n,m)_i \cong J(n,n-m)_i$ (and an 'outer automorphism' of the graph when $m = n/2$), so we may assume without loss of generality that $m \leq n/2$. We will also assume that $m \geq 2$ (so $n \geq 4$) since the regular embeddings of complete graphs K_n have been completely classified (see Section 5).

We now consider the connectedness of the graphs $J(n,m)_i$. Since $J(n,m)_0$ is a null graph, we will assume here that $i > 0$. In the action of S_n on Ω, the stabiliser of an m-element subset $M \in \Omega$ has the form $S_m \times S_{n-m}$, where the direct factors act naturally on M and $\overline{M}$; for $m \neq n/2$ this stabiliser is a maximal subgroup of S_n (see, for instance, [8, Exercise 5.2.8]), so S_n acts primitively on Ω and hence $J(n,m)_i$ is connected for each $i = 1,\ldots,m$ ([8, Theorem 3.2A] and [27]). If $m = n/2$, however, there is a unique group between the stabiliser $S_m \times S_m$ and S_n, namely the wreath product $S_m \wr S_2$, a semidirect product $(S_m \times S_m) : S_2$ of $S_m \times S_m$ by S_2, stabilising the partition $\{M, \overline{M}\}$ of N; thus S_n is imprimitive on Ω, permuting the $e = d/2$ complementary pairs M and $\overline{M}$. In this case, if we choose $g \in S_n$ sending M to some $M' \in \Gamma_i(M)$, then $g \in S_m \wr S_2$ if and only if $i = 0$ or m; it follows that if $1 \leq i \leq m-1$ then $\langle S_m \wr S_2, g\rangle = S_n$ so a result of Glauberman [14] implies that $J(2m,m)_i$ is connected. On the other hand, $J(2m,m)_m$ consists of e disjoint copies of K_2, one for each pair $\{M, \overline{M}\}$.

If I is any subset of $\{1,\ldots,m\}$ we define the *merged Johnson graph* $J(n,m)_I$ to be the edge-union of the graphs $J(n,m)_i$ where $i \in I$. Thus $J(n,m)_I$ has vertex-set $\Omega = N^{\{m\}}$, with vertices M and M' adjacent if and only if $|M \cap M'| = m - i$ for some $i \in I$. We will denote the set of neighbours of M in $J(n,m)_I$ by $\Gamma_I(M) = \bigcup_{i\in I}\Gamma_i(M)$. The preceding comments about the connectedness of $J(n,m)_i$ show that with the exceptions of $J(n,m)_\emptyset$ and $J(2m,m)_m$, the graphs $J(n,m)_I$ are all connected.

3. Automorphism groups of the merged Johnson graphs

Before considering regular embeddings, we need first to determine the automorphism group of each graph $J(n,m)_I$. In many cases this is already in the literature: for instance, Whitney's theorem [32] on automorphisms of line graphs deals with the case $J(n,2)_1 = L(K_n)$, and further examples are given in [3, Section 9.1], [10] and [24]. Nevertheless, it is useful to state and prove the full result here. It can be proved by a careful examination of the structure constants of the Johnson association scheme, and indeed this has been done in certain cases by Klin [24]. However, it is more convenient for us to use the classification of the overgroups of S_n in S_d; this was begun by Kalužnin and Klin [22], developed further by Halberstadt [16], and completed by Ustimenko-Bakumovskiĭ, who summarised his results in [29] and gave full details in [30]. Skalba [28] has also published a proof (like Halberstadt, avoiding the hardest case $m = n/2$), while Faradžev, Klin and Muzichuk have provided a useful algebraic and combinatorial overview of this topic in [12, Section 3.2].

Since S_n, acting naturally on the n-element set N, has a faithful induced action on the set $\Omega = N^{\{m\}}$ for each $m = 1, \ldots, n-1$, we will abuse the notation by identifying S_n with its image in the symmetric group S_d of all permutations of Ω, where $d = |\Omega| = \binom{n}{m}$. By considering the effect on Ω of transposing two elements of N, we see that S_n is contained in the alternating group A_d if and only if $\binom{n-2}{m-1}$ is even. Any additional overgroups of S_n in S_d are given by the following result of Ustimenko-Bakumovskiĭ:

Proposition 1 ([29, 30]). *The group S_n, acting on Ω for $2 \le m \le n/2$, is maximal in S_d or A_d if $(n, m) \ne (6, 2), (8, 2), (10, 3), (12, 4), (2m + 1, m)$ or $(2m, m)$. For these exceptional pairs (n, m), the overgroups of S_n other than S_d and A_d are as follows:*

$$\begin{aligned}
(6,2)&: \; S_6 < GL_4(2) && \text{with } d = 15,\\
(8,2)&: \; S_8 < Sp_6(2) && \text{with } d = 28,\\
(10,3)&: \; S_{10} < Sp_8(2) && \text{with } d = 120,\\
(12,4)&: \; S_{12} < O^-_{10}(2) && \text{with } d = 495,\\
(5,2)&: \; S_5 < S_6 < \operatorname{Aut} S_6 && \text{with } d = 10,\\
(2m+1,m)&: \; S_n < S_{n+1} && \text{for } m \ge 3,\\
(2m,m)&: \; S_n < \text{various imprimitive subgroups of } S_2 \wr S_e, && \text{where } e = d/2.
\end{aligned}$$

In the first six cases, where $n > 2m$, there is a unique overgroup of S_n of each type listed. In the last case, where $n = 2m$, the overgroups are explicitly described in [29]; here we simply need the fact that they are imprimitive, permuting complementary pairs M and $\overline{M}$, as proved in [29]. To avoid confusion, note that when $(n, m) = (12, 4)$ the overgroup is not the simple orthogonal group G of order $2^{20}.3^6.5^2.7.11.17$, but rather its automorphism group, of twice this order, denoted by $G.2 = GO^-_{10}(2)$ in the ATLAS [5, p. 147]; we have followed [9, 12, 28, 29] in using the classical notation $O^-_{10}(2)$ for this overgroup, whereas in the ATLAS and in [8, Appendix B] this notation indicates its simple subgroup of index 2, which contains A_{12} but not S_{12}. The remarks in [28, pp. 159, 172] that this overgroup is simple appear to be based on a misunderstanding. For a detailed description of this and many other embeddings of symmetric groups in orthogonal and symplectic groups, see [9].

For notational simplicity, we write $J = J(n, m)_I$ and $A = \operatorname{Aut} J$. Since $J = \bigcup_{i \in I} J(n, m)_i$ we have $\bigcap_{i \in I} \operatorname{Aut} J(n, m)_i \le A$. If $I = \emptyset$ or $\{1, \ldots, m\}$ then J is a null or complete graph, so $A = S_d$; we will therefore assume that $\emptyset \subset I \subset \{1, \ldots, m\}$. We define $I' = I \backslash \{m\}$, and for any integer k we define $k - I = \{k - i \mid i \in I\}$, with a similar definition of $k - I'$. We also define $I'' = m - I'$. If $m = n/2$ we define $e = (1/2)\binom{n}{m}$.

Theorem 2. *Let $J = J(n, m)_I$ where $2 \le m \le n/2$ and $\emptyset \subset I \subset \{1, \ldots, m\}$, and let $A = \operatorname{Aut} J$.*

(a) *If* $2 \leq m < (n-1)/2$, *and* $J \neq J(12,4)_I$ *with* $I = \{1,3\}$ *or* $\{2,4\}$, *then* $A = S_n$*; this group has rank* $1+m$ *on the vertex-set* Ω *of* J, *with orbitals* $\Gamma_0, \ldots, \Gamma_m \subset \Omega^2$.
(b) *If* $(n,m) = (12,4)$ *with* $I = \{1,3\}$ *or* $\{2,4\}$, *then* $A = O^-_{10}(2)$ *with orbitals* $\Gamma_0, \Gamma_1 \cup \Gamma_3$ *and* $\Gamma_2 \cup \Gamma_4$.
(c) *If* $m = (n-1)/2$ *and* $I \neq m+1-I$, *then* $A = S_n$ *with orbitals* $\Gamma_0, \ldots, \Gamma_m$.
(d) *If* $m = (n-1)/2$ *and* $I = m+1-I$, *then* $A = S_{n+1}$ *with orbitals* Γ_0 *and* $\Gamma_i \cup \Gamma_{m+1-i}$ *for* $i = 1, \ldots, \lfloor (m+1)/2 \rfloor$.
(e) *If* $m = n/2$ *and* $I \neq \{m\}$ *or* $\{1, \ldots, m-1\}$, *and* $I' \neq I''$, *then* $A = S_2 \times S_n$ *with orbitals* $\Gamma_0, \ldots, \Gamma_m$.
(f) *If* $m = n/2$ *and* $I \neq \{m\}$ *or* $\{1, \ldots, m-1\}$, *and* $I' = I''$, *then* $A = S_2^e : S_n$ *with orbitals* $\Gamma_0, \Gamma_i \cup \Gamma_{m-i}$ *for* $i = 1, \ldots, \lfloor m/2 \rfloor$, *and* Γ_m.
(g) *If* $m = n/2$ *and* $I = \{1, \ldots, m-1\}$ *or* $\{m\}$, *then* $A = S_2^e : S_e = S_2 \wr S_e$ *with orbitals* $\Gamma_0, \Gamma_1 \cup \cdots \cup \Gamma_{m-1}$ *and* Γ_m.

[The precise nature and action of each group A will be explained within the proof.]

Proof. The natural actions of S_n on N and of A on Ω induce inclusions $S_n \leq A \leq S_d$, where $d = \binom{n}{m}$. The graph J is neither complete nor null, so A cannot be doubly transitive, and hence $A \ngeq A_d$ since $d \geq 4$. For most pairs n and m (see Proposition 1 for details), S_n is a maximal subgroup of either S_d or A_d, so it follows in these cases that $A = S_n$, a rank $1+m$ group on Ω with orbitals $\Gamma_0, \ldots, \Gamma_m$. We now consider in more detail when this argument applies, and when additional arguments are required.

We first consider cases (a) and (b) of the theorem, where $2 \leq m < (n-1)/2$, so that $n \geq 6$. By Proposition 1, the preceding argument deals with all cases where $m \geq 5$; it also deals with the case $m = 4$ provided $n \neq 12$, with $m = 3$ provided $n \neq 10$, and with $m = 2$ provided $n \neq 6$ or 8. When $(n,m) = (6,2)$, $(8,2)$ or $(10,3)$, the overgroups $GL_4(2)$, $Sp_6(2)$ and $Sp_8(2)$ are all doubly transitive on Ω, so the same argument gives $A = S_n$. When $(n,m) = (12,4)$, however, the overgroup $O^-_{10}(2)$ has rank 3; its orbitals are Γ_0, $\Gamma_1 \cup \Gamma_3$ and $\Gamma_2 \cup \Gamma_4$ (see [3, p. 261] or [12, p. 101]), so $J(12,4)_I$ is invariant under $O^-_{10}(2)$ only for $I = \{1,3\}$ and $\{2,4\}$. Thus A is $O^-_{10}(2)$ in these two cases, and S_{12} otherwise.

This deals with cases (a) and (b) of the Theorem. To understand cases (c) and (d), where $m = (n-1)/2$, we need to explain the action of the overgroup S_{n+1} on Ω. Let $N^* = N \cup \{\infty\}$ for some symbol $\infty \notin N$, so $|N^*| = n+1 = 2(m+1)$, and let Φ be the set of *equipartitions* of N^*, by which we mean the unordered partitions $\{P_1, P_2\}$ of N^* with $|P_1| = |P_2|$, so $|P_j| = m+1$ for each j. There is a bijection $\beta : \Omega \to \Phi$, sending each $M \in \Omega$ to the equipartition $\{M \cup \{\infty\}, N \backslash M\}$; its inverse sends each $\{P_1, P_2\} \in \Phi$ to $M = P_j \backslash \{\infty\}$, where j is chosen so that $\infty \in P_j$. The natural action of S_{n+1} on N^* induces an action on Φ and hence, via β, on Ω; the restriction of this action to the subgroup S_n of S_{n+1} fixing ∞ agrees with the original action of S_n on Ω, so we obtain the inclusions $S_n < S_{n+1} < S_d$. When $n = 5$, so that $d = \binom{5}{2} = 10$, we can identify this overgroup S_6 with $P\Sigma L_2(9)$, acting naturally on the 10 points of the projective line over $GF(9)$, and this is contained in an additional overgroup $P\Gamma L_2(9) \cong \mathrm{Aut}\, S_6$ of S_5 in S_{10}.

In its action on Φ (and hence on Ω), S_{n+1} has rank $1 + \lfloor (m+1)/2 \rfloor$: for each $i = 0, \ldots, \lfloor (m+1)/2 \rfloor$, it has an orbit on Φ^2 consisting of those pairs of equipartitions $(\{P_1, P_2\}, \{P'_1, P'_2\})$ of N^* such that each of the four intersections $P_j \cap P'_k$ has size i or $m+1-i$. Deleting ∞ from whichever sets P_j and P'_k contain it, we see that these pairs of equipartitions correspond under β to the pairs $(M, M') \in \Omega^2$ where $|M \cap M'| = i - 1$ or $m - i$. For $i > 0$ these therefore form an orbit $\Delta_i = \Gamma_{m+1-i} \cup \Gamma_i$ of S_{n+1} on Ω^2, a union of two distinct orbits of S_n unless $i = (m+1)/2$ with m odd. With this one exception, the orbitals $\Gamma_1, \ldots, \Gamma_m$ of S_n are thus merged in pairs under the action of S_{n+1}. Since all the overgroups of S_n apart from S_{n+1} are doubly transitive on Ω, it follows that $A = S_n$ unless $I = m + 1 - I$, in which case $A = S_{n+1}$. This deals with cases (c) and (d).

To deal with the remaining cases, let $m = n/2$. Since A is an overgroup of S_n in S_d, not containing A_d, it follows from Proposition 1 that A is an imprimitive subgroup of $S_2 \wr S_e$, permuting the set Φ consisting of the $e = d/2$ equipartitions $E = \{P, \overline{P}\}$ of N. First we determine the kernel $C = A \cap B$ of the action of A on Φ, where B is the base group S_2^e of $S_2 \wr S_e$. Each direct factor S_2 of B is generated by a permutation t_E ($E \in \Phi$) of Ω which transposes the parts P and $\overline{P}$ of E, while fixing all other elements of Ω. For any $\Psi \subseteq \Phi$, let $t_\Psi = \prod_{E \in \Psi} t_E$, so $t_\Psi \mapsto \Psi$ is an isomorphism between B and the power set of Φ. In particular, let D denote the diagonal subgroup $\langle t_\Phi \rangle \cong S_2$ of B, where t_Φ sends every element of Ω to its complement. For each i, if $(P, Q) \in \Gamma_i$ then $|\overline{P} \cap \overline{Q}| = |\overline{P \cup Q}| = 2m - (m + i) = m - i$, so $(\overline{P}, \overline{Q}) \in \Gamma_i$. Thus $t_\Phi \in \bigcap_{i \in I} \operatorname{Aut} J(n, m)_i \leq A$ and hence A contains $\langle D, S_n \rangle = D \times S_n \cong S_2 \times S_n$.

Suppose first that $I' \neq I''(:=m - I')$, say $i \in I$ but $m - i \notin I$ for some $i \neq m$. Let $t_\Psi \in A$ for some $\Psi \neq \emptyset$, say $E = \{P, \overline{P}\} \in \Psi$. If $E' = \{Q, \overline{Q}\} \in \Phi$ where $(P, Q) \in \Gamma_i$ then since $|\overline{P} \cap \overline{Q}| = m - i$ whereas $|\overline{P} \cap Q| = i$ it follows that t_Ψ must send Q to $\overline{Q}$; thus $E' \in \Psi$ for all such E', and since $J(n, m)_i$ is connected it follows by iterating this argument that $\Psi = \Phi$. Conversely $t_\Phi \in A$, so this shows that if $I' \neq I''$ then $C = D$. When $I' = I''$, however, we have $t_E \in A$ for each $E \in \Phi$, so $C = B$.

We now consider the permutation group $S = A/C$ induced by A on Φ. First let $n > 4$, so the subgroup S_n of A acts faithfully on Φ, and hence S is an overgroup of S_n in S_e. As in cases (c) and (d), but now replacing n and m with $n-1$ and $m-1$, by choosing an element $\infty \in N$ we obtain a bijection between the equipartitions $\{P_1, P_2\} \in \Phi$ of N and the $(m-1)$-element subsets $P_j \backslash \{\infty\}$ of $N_0 = N \backslash \{\infty\}$, where $\infty \in P_j$. The symmetric groups on N_0 and N then give rise to inclusions $S_{n-1} < S_n < S_e$, where $e = (1/2)\binom{n}{m} = \binom{n-1}{m-1}$. By applying Proposition 1 to the overgroups of S_{n-1} in S_e we see from the inclusions $S_n \leq S \leq S_e$ that S must be S_n, A_e or S_e, or possibly $\operatorname{Aut} S_6 = P\Gamma L_2(9)$ if $n = 6$ (so $e = 10$). Apart from S_n for $n \geq 8$, these groups are all doubly transitive on Φ, so if $S > S_n$ then the stabiliser A_E in A of an equipartition $E = \{M, \overline{M}\} \in \Phi$ permutes the $e - 1$ equipartitions $E' \in \Phi \backslash \{E\}$ transitively. Now $A_E = D \times A_M$, with D acting trivially on Φ, so A_M is also transitive on $\Phi \backslash \{E\}$ and hence has at most two orbits on $\Omega \backslash \{M, \overline{M}\}$.

If A_M is transitive on $\Omega \backslash \{M, \overline{M}\}$, then A has rank 3, with orbitals Γ_0, $\Gamma_1 \cup \cdots \cup \Gamma_{m-1}$ and Γ_m; thus $I = \{1, \ldots, m-1\}$ or $\{m\}$, so J is the multipartite graph $K_{2,\ldots,2}$ or its complement (e disjoint copies of K_2), and A is the semidirect product $B : S_e = S_2 \wr S_e$ as in case (g). If A_M has two orbits on $\Omega \backslash \{M, \overline{M}\}$, they are transposed by D and hence each contains the complements of the sets in the other; thus A has rank 4, and $\Gamma_i(M)$ and $\Gamma_{m-i}(M)$ are contained in different orbits of A_M for each i, so $\{1, \ldots, m-1\}$ is partitioned by I' and I''. This means that $J/D \cong K_e$, so J is an antipodal double cover of K_e. Each triangle $K_3 \subset K_e$ lifts to either a disjoint pair of triangles or a cycle of length 6 in J, and since the groups $S > S_n$ listed above are all triply transitive on Φ, every K_3 lifts in the same way. If they lift to pairs of triangles then J consists of two disjoint copies of K_e, and if they lift to cycles, then J is the cocktail party graph $K_e \otimes K_2$, that is, the complete bipartite graph $K_{e,e}$ minus a matching [26]. In either case, A acts imprimitively on Ω, with two blocks of size e, and hence so does its subgroup S_n. These blocks must be the orbits of a subgroup of index 2 in S_n, whereas the only such subgroup is A_n, which acts transitively. Thus, unless we are in case (g) we have $S = S_n$, so $A = C : S_n$; it then follows from our earlier investigation of C that $A = D \times S_n$ or $B : S_n$ as $I' \neq I''$ or $I' = I''$, giving cases (e) and (f). The orbitals $\Gamma_0, \ldots, \Gamma_m$ of S_n are preserved by D, while B preserves Γ_0 and Γ_m and transposes the pairs Γ_i and Γ_{m-i} $(i = 1, \ldots, \lfloor m/2 \rfloor)$, so in each case the orbitals of A are as stated in the theorem.

We earlier excluded the case $n = 2m = 4$; here $I = \{1\}$ or $\{2\}$, and J is either the octahedral graph $J(4,2)_1 = J(4,2) = L(K_4)$ or its complement $J(4,2)_2$, three disjoint copies of K_2. In either case, it is easily seen that $A = S_2^3 : S_3 = S_2 \wr S_3$, as in case (g). □

Comments. (1) As noted in [3, p. 261] and [12, p. 101], the merging of orbitals Γ_i in case (b) of Theorem 2 yields a strongly regular graph on 495 vertices. One can deduce this merging in several ways. One is by considering the actions of S_{12} and $O_{10}^-(2)$ on the 10-dimensional binary vector space consisting of the even order subsets $M \subseteq N$ modulo complementation, preserving the quadratic form $(1/2)|M| \bmod (2)$, with the 4-element subsets $M \in \Omega$ corresponding to the non-zero isotropic vectors. Alternatively, the irreducible constituents of the permutation character of $O_{10}^-(2)$ have degrees 1, 154 and 340 [5, p. 147], and the subdegrees $|\Gamma_i(M)|$ for S_{12} on Ω are 1, 32, 168, 224 and 70 for $i = 0, \ldots, 4$; the only rank 3 merging of these subdegrees which satisfies Frame's criterion (see [3, Theorem 2.2.4], [13] or [33, Theorem 30.1(A)]) is 1, $32 + 224 = 256$, $168 + 70 = 238$, so the orbitals of A are Γ_0, $\Gamma_1 \cup \Gamma_3$ and $\Gamma_2 \cup \Gamma_4$.

(2) In the proof for case (d), the bijection β can be used to identify $J(n,m)_I = J(2m+1,m)_I$ with the graph whose vertices are the equipartitions of the $(n+1)$-element set N^*, adjacent if their parts intersect in i-element sets for some $i \in I$ (well-defined since $I = m + 1 - I$). This can be regarded as the distance I graph $\overline{J}(2m+2, m+1)_I$ of the folded Johnson graph (or *even graph*) $\overline{J}(2m+2, m+1) = E_{m+1}$ formed from $J(2m+2, m+1)$ by identifying every $(m+1)$-element subset of N^* with its complement; it has automorphism group $S_{n+1} = S_{2m+2}$ acting naturally on N^* (see [3, Section 9.1C] or [17, Section 6.2] for details of the even graphs).

4. Preliminary results on permutation groups

In order to apply Theorem 2 to the regular embeddings of the merged Johnson graphs, we need some further concepts and results on permutation groups. See [8] for full details.

A permutation group G, acting on a set N, is *m-homogeneous* if its induced action on $N^{\{m\}}$ is transitive. The following result [8, Theorem 9.4A] is originally due to Brown [4]:

Proposition 3 ([4]). *Let G act on a set N of n elements. If $0 \leq l \leq m$ and $l + m \leq n$, then G has at least as many orbits on $N^{\{m\}}$ as it has on $N^{\{l\}}$. If G is m-homogeneous where $0 < 2m \leq n + 1$ then G is l-homogeneous for all l with $0 < l \leq m$; in particular, G is transitive on N.*

The next result [8, Theorem 9.4B] is by Livingstone and Wagner [25] and Kantor [23]:

Proposition 4 ([23, 25]). *Let G be an m-homogeneous group of degree n, where $3 \le m \le n/2$. Then G is $(m-1)$-transitive, and with the following exceptions G is m-transitive:*

(a) $m = 4$ *and* $G = PGL_2(8)$ *or* $P\Gamma L_2(8)$ *with* $n = 9$, *or* $G = P\Gamma L_2(32)$ *with* $n = 33$.
(b) $m = 3$ *and* $PSL_2(q) \le G \le P\Sigma L_2(q)$ *for* $q \equiv 3 \bmod (4)$ *with* $n = q + 1$, *or* $G = AGL_1(8)$ *or* $A\Gamma L_1(8)$ *with* $n = 8$, *or* $G = A\Gamma L_1(32)$ *with* $n = 32$.

There is a similar result for $m = 2$, but we do not need it here. We also need the following results about cyclic groups; the proofs, which are straightforward, are omitted.

Lemma 5. *Let C be a cyclic permutation group of degree n. If C is m-homogenous where $0 < m < n$, then C is transitive and $m = 1$ or $n - 1$.*

Lemma 6. *Let C be a cyclic group, acting on finite sets Ω_1 and Ω_2. Then the following are equivalent:*

(a) *C is transitive in its induced action on $\Omega_1 \times \Omega_2$;*
(b) *C is transitive on Ω_1 and Ω_2, and $|\Omega_1|$ is coprime to $|\Omega_2|$.*

A k-transitive permutation group is *sharply k-transitive* if the stabiliser of k points is the identity subgroup [8, Section 7.6]. Zassenhaus [35] has shown that there are just two families of sharply 3-transitive finite permutation groups, both of them subgroups of $P\Gamma L_2(q)$ acting with degree $n = q + 1$ on the projective line over $GF(q)$. One such subgroup is $PGL_2(q)$, consisting of the Möbius transformations

$$z \mapsto \frac{az+b}{cz+d} \qquad (a, \ldots, d \in GF(q), ad - bc \neq 0). \tag{1}$$

If the prime power q is an odd square there is a second sharply 3-transitive subgroup $M_2(q)$, consisting of the transformations (1) for which $ad - bc$ is a square, together with the transformations

$$z \mapsto \frac{a\overline{z}+b}{c\overline{z}+d} \qquad (a, \ldots, d \in GF(q))$$

for which $ad - bc$ is a non-square, where $z \mapsto \overline{z} = z^{\sqrt{q}}$ is the automorphism of order 2 of $GF(q)$. The smallest example $M_2(9)$ of this family is M_{10}, the stabiliser of a point in the simple Mathieu group M_{11} of degree 11 [5].

Finally, we need a purely number-theoretic result:

Proposition 7. *If $2 \le m \le n/2$, the binomial coefficient $\binom{n}{m}$ is not a prime power.*

Proof. This is elementary for $m = 2$ and 3, and otherwise it follows from a theorem of Erdős [11] that $\binom{n}{m}$ is not a proper power for $4 \le m \le n - 4$ (see also [1, Chapter 3]). □

5. Regular maps

Coxeter and Moser [7, Chapter 8] define a map on a surface to be *regular* if it has automorphisms cyclically permuting the successive edges around a face and around an incident vertex. If $\mathcal{M}$ is an orientable regular map then its orientation-preserving automorphism group $G = \mathrm{Aut}^+\mathcal{M}$ acts regularly on the directed edges of $\mathcal{M}$. In particular, G acts transitively on the vertices, and moreover the stabiliser of a vertex is a cyclic group, acting regularly on the incident edges. Here $|G| = 2E$ where E is the number of edges of $\mathcal{M}$, whereas in the case of a non-orientable regular map, the automorphism group $G = \mathrm{Aut}\,\mathcal{M}$ has order $4E$: again G acts transitively on the vertices, but now their stabilisers are dihedral groups acting naturally on the incident edges.

A map has *type* $\{p, q\}$ if its faces are p-gons and its vertices have valency q. In [6] and [7], the notation $\{p, q\}$ also denotes the unique simply connected map of type $\{p, q\}$; this is drawn on the sphere, the Euclidean plane or the hyperbolic plane as $p^{-1} + q^{-1} > 1/2, = 1/2$ or $< 1/2$, and it covers all other maps of this type. The orientation-preserving automorphism group of $\{p, q\}$ is the triangle group

$$\Delta(q, 2, p) = \langle X, Y, Z \mid X^q = Y^2 = Z^p = XYZ = 1\rangle,$$

where X, Y and Z are rotations of the map through $2\pi/q$, π and $2\pi/p$ around a vertex and the centres of an incident edge and face. The orientable regular maps of type $\{p, q\}$ are the quotients $\mathcal{M} = \{p, q\}/K$, where K is a torsion-free normal subgroup of $\Delta(q, 2, p)$; up to isomorphism, there is one map for each such K, and its orientation-preserving automorphism group $\mathrm{Aut}^+\mathcal{M}$ is isomorphic to $\Delta(q, 2, p)/K$.

The full automorphism group of $\{p, q\}$ is the extended triangle group

$$\Delta[q, 2, p] = \langle R_0, R_1, R_2 \mid R_i^2 = (R_1R_2)^q = (R_2R_0)^2 = (R_0R_1)^p = 1\rangle,$$

where R_0, R_1 and R_2 are reflections preserving an incident edge and face, face and vertex, and vertex and edge, so that $X = R_1R_2$, $Y = R_2R_0$ and $Z = R_0R_1$ generate the even subgroup $\Delta(q, 2, p)$ of index 2 in $\Delta[q, 2, p]$. An orientable regular map $\mathcal{M}$ is reflexible (has orientation-reversing automorphisms) if and only if K is normal in $\Delta[q, 2, p]$, in which case its full automorphism group $\mathrm{Aut}\,\mathcal{M}$ is isomorphic to $\Delta[q, 2, p]/K$. Non-orientable regular maps of type $\{p, q\}$ have the form $\mathcal{M} \cong \{p, q\}/K$ for normal subgroups K of $\Delta[q, 2, p]$ not contained in $\Delta(q, 2, p)$, with $\mathrm{Aut}\,\mathcal{M} \cong \Delta[q, 2, p]/K$.

We refer to [7, Chapter 8] for further background and notation for regular maps, and to [20] for their connections with triangle groups. In particular, $\{p, q\}_r$ denotes the map $\{p, q\}/K$ where K is the normal closure of $(R_0R_1R_2)^r$, so that the Petrie polygons (closed zig-zag paths) have length r; this map is orientable if and only if r is even. The *Petrie dual* of a map embeds the same graph, but has faces and Petrie polygons transposed, the two maps having the same automorphism group [7, Sections 5.2 and 8.6]; this map operation, along with others, is considered in more detail in [20, 21, 34]. The notation $\{5, 5/2\}$ denotes the great dodecahedron, an orientable regular map of type $\{5, 5\}$ and genus 4 [6, Section 6.2]; it embeds the graph of the icosahedron $\{3, 5\}$ with 12 pentagonal faces, each spanning the neighbours of a vertex, and it has the same automorphism group $S_2 \times A_5$ as $\{3, 5\}$, with the factor S_2 generated by the antipodal automorphism.

Before considering the regular embeddings of merged Johnson graphs, we need some results of Biggs [2] and James [18] on regular embeddings of complete graphs.

Proposition 8 ([2]). *The complete graph K_d has an orientable regular embedding if and only if d is a prime power.*

In fact, it is shown in [19] that the only orientable regular embeddings of K_d are those constructed by Biggs, as Cayley maps over the additive group of the field $GF(d)$; they correspond to the orbits of the Galois group of the field on generators of its multiplicative group, so there are $\phi(d-1)/e$ such maps, where $d = p^e$ for some prime p and ϕ denotes Euler's function.

Proposition 9 ([18]). *The only non-orientable regular embeddings of a complete graph are the antipodal quotients of a dihedron* $\{6, 2\}$ *for* K_3*, of a cube* $\{4, 3\}$ *for* K_4*, and of an icosahedron* $\{3, 5\}$ *and a great dodecahedron* $\{5, 5/2\}$ *for* K_6*. The first three maps have genus 1, and the fourth has genus 5.*

The embeddings in Proposition 9 are isomorphic to the maps $\{6, 2\}_3$, $\{4, 3\}_3$, $\{3, 5\}_5$ and $\{5, 5\}_3$ discussed in [7, Section 8.6 and Table 8]. The first two maps are the Petrie duals of the orientable regular embeddings $\{3, 2\}$ and $\{3, 3\}$ of K_3 and K_4 (a dihedron and a tetrahedron); they have automorphism groups $D_6 \cong S_2 \times S_3$ and S_4. The last two maps are Petrie duals of each other, both having automorphism group $PSL_2(5) \cong A_5$.

6. Regular embeddings of merged Johnson graphs

We now classify the regular embeddings of the graphs $J = J(n, m)_I$; they are described in more detail in Section 8. To ensure that J is connected, we will assume that $I \neq \emptyset$, and also that $I \neq \{m\}$ if $m = n/2$. We denote an orientable or non-orientable compact connected surface of genus g by $\mathcal{S}_g^+$ or $\mathcal{S}_g^-$ respectively: for instance, $\mathcal{S}_0^+$ and $\mathcal{S}_1^-$ are the sphere and the real projective plane.

Theorem 10. *Let* $J = J(n, m)_I$*, where* $2 \le m \le n/2$ *and* $\emptyset \subset I \subseteq \{1, \ldots, m\}$*, and* $I \neq \{m\}$ *if* $m = n/2$*. Then* J *has only the following regular embeddings:*

(a) *the octahedron* $\{3, 4\}$ *and its Petrie dual* $\{6, 4\}_3$*, which embed* $J(4, 2)_1 = L(K_4)$ *in* $\mathcal{S}_0^+$ *and* $\mathcal{S}_4^-$ *with automorphism group* $S_2 \wr S_3 \cong S_2 \times S_4$*;*
(b) *the embeddings* $\{3, 5\}_5$ *and* $\{5, 5\}_3$ *of* $J(4, 2)_{1,2} = K_6$ *in* $\mathcal{S}_1^-$ *and* $\mathcal{S}_5^-$*, which are a Petrie dual pair of maps arising as the antipodal quotients of an icosahedron* $\{3, 5\}$ *and a great dodecahedron* $\{5, 5/2\}$*, with automorphism group* $PSL_2(5) \cong A_5$*;*
(c) *a Petrie dual pair of embeddings of type* $\{4, 6\}$ *and* $\{5, 6\}$ *of* $J(5, 2)_1 = L(K_5)$ *in* $\mathcal{S}_7^-$ *and* $\mathcal{S}_{10}^-$*, which are quotients of* $\{4, 6\}_5$ *and* $\{5, 6\}_4$ *by a central involution, with automorphism group* S_5*;*
(d) *the embedding* $\{5, 3\}_5$ *of Petersen's graph* $J(5, 2)_2 = \overline{L(K_5)}$ *in* $\mathcal{S}_1^-$*, arising as the antipodal quotient of a dodecahedron* $\{5, 3\}$*, with automorphism group* A_5*.*

Proof. We use case-by-case analysis, considering the possibilities for $A = \operatorname{Aut} J$ in Theorem 2. First we eliminate a case not covered there. If $I = \{1, \ldots, m\}$ then $J = K_d$. By Proposition 8, K_d has an orientable regular embedding if and only if d is a prime power, and by Proposition 7 this is never the case if d is a binomial coefficient $\binom{n}{m}$ with $2 \le m \le n/2$. If K_d has a non-orientable regular embedding then Proposition 9 implies that $d \le 6$; since $d = \binom{n}{m}$ with $2 \le m \le n/2$, only the case $d = 6 = \binom{4}{2}$ arises here, and Proposition 9 gives the embeddings described in part (b) of the theorem. We may therefore assume from now on that $I \subset \{1, \ldots, m\}$, as in Theorem 2.

Let $\mathcal{M}$ be a regular embedding of J, and G its automorphism group (orientation-preserving if $\mathcal{M}$ is orientable). Since G acts faithfully on J, it is a subgroup of A acting transitively on the directed edges, with the stabiliser of a directed edge having order $\eta = 1$ or 2 as $\mathcal{M}$ is orientable or not. These directed edges can be identified with the pairs $(M, M') \in \Gamma_I = \cup_{i \in I} \Gamma_i$, so

$$|G| = \eta|\Gamma_I| = \eta \sum_{i \in I} \binom{n}{m}\binom{m}{i}\binom{n-m}{i} = \eta \sum_{i \in I} \frac{n!}{i!^2(m-i)!(n-m-i)!}. \tag{2}$$

By Theorem 2, there is a subgroup S_n of A consisting of the automorphisms of J induced by the permutations of N.

Cases (a), (c). Suppose that $G \leq S_n$, as must happen in cases (a) and (c) of Theorem 2, and may happen in other cases, so we can study G through its action on N. Since S_n has orbitals $\Gamma_0, \ldots, \Gamma_m \subset \Omega^2$, it follows that if $|I| > 1$ then G cannot be transitive on Γ_I, contradicting the regularity of $\mathcal{M}$. Hence $|I| = 1$, so $J = J(n, m)_i$ for some $i = 1, \ldots, m$.

The stabiliser G_M in G of each vertex $M \in \Omega$ of $\mathcal{M}$ is a cyclic or dihedral group, permuting the neighbours of M in its natural representation, so it contains a cyclic subgroup G^*_M of index $\eta = 1$ or 2 permuting the neighbours regularly. The setwise stabiliser G_M of each m-element subset $M \subset N$ therefore has a cyclic subgroup G^*_M of index η acting regularly on those m-element subsets $M' \subset N$ such that $|M \cap M'| = m - i$. These are the sets $M' = (M \backslash I_1) \cup I_2$, where I_1 and I_2 are i-element subsets of M and $\overline{M}$, so G^*_M acts regularly on the ordered pairs (I_1, I_2) of such subsets. By Lemma 6, it follows that the cyclic group G^*_M induces transitive groups G_1 and G_2 of coprime orders on the i-element subsets of M and of $\overline{M}$. By Lemma 5, this transitivity condition on the cyclic group G_1 forces $i = 0, 1, m-1$ or m, and the condition on G_2 forces $i = 0, 1, n-m-1$ or $n-m$. Since we are assuming that $i \neq 0$, the only possibilities are therefore

(i) $i = 1$, or
(ii) $i = m - 1 = n - m - 1$, or
(iii) $i = m - 1 = n - m$, or
(iv) $i = m = n - m - 1$, or
(v) $i = m = n - m$.

In case (i), G_1 and G_2 are regular cyclic groups of orders m and $n - m$; since these orders are coprime, m and n are coprime. In (ii), $n = 2m$ and $i = m - 1$, so G_1 and G_2 are both regular cyclic groups of order m, contradicting the coprimality of their orders. In (iii) we have $n = 2m - 1$, contradicting our assumption that $m \leq n/2$. We have excluded case (v) since $J(2m, m)_m$ is not connected for $m \geq 2$. This leaves case (i), where J is a Johnson graph $J(n, m)$ with $\gcd(n, m) = 1$, and case (iv), where $n = 2m + 1$ and $i = m$, so that $J = J(2m + 1, m)_m$ is an odd graph O_{m+1}. In cases (i) and (iv), Eq. (2) gives

$$|G| = \eta\binom{n}{m}m(n-m) = \frac{\eta n!}{(m-1)!(n-m-1)!}, \tag{3}$$

and

$$|G| = \eta\binom{n}{m}(m+1) = \frac{\eta(2m+1)!}{m!^2}. \tag{4}$$

A second consequence of the regularity of $\mathcal{M}$ is that G acts transitively on the vertices of J, that is, on the m-element subsets of N, so it acts on N as an m-homogeneous permutation group. Since $m \le n/2$, Proposition 4 implies that G is m-transitive if $m \ge 5$; the set-wise stabiliser G_M of an m-element set $M \subset N$ then acts on M as S_m, and hence cannot be cyclic or dihedral, so $m \le 4$. We now consider such values of m in turn.

If $m = 4$ then Proposition 4 shows that G is 4-transitive, giving a similar contradiction, unless $G = PGL_2(8)$ or $P\Gamma L_2(8)$ with $n = 9$, or $G = P\Gamma L_2(32)$ with $n = 33$. These groups have orders 9.8.7, 9.8.7.3 and 33.32.31.5, whereas by Eq. (3) the groups G in case (i) have orders $9.8.7.5\eta$ and $33.32.31.5.29\eta$ for $(n, m) = (9, 4)$ and $(33, 4)$; similarly in case (iv), where $(n, m) = (9, 4)$, Eq. (4) gives $|G| = 9.7.5.2\eta$, so these exceptions do not arise.

Now let $m = 3$. In case (i) we have $|G| = n(n-1)(n-2)(n-3)\eta/2$. Since G is transitive on adjacent pairs (M, M') it is transitive on 4-element subsets $M \cup M' \subset N$, that is, 4-homogeneous on N. Hence G is 3-transitive by Proposition 4, leading to a similar contradiction in the orientable case. In the non-orientable case, where $\eta = 2$, G is 4-transitive since the exceptions in Proposition 4 have the wrong orders, and hence G is sharply 4-transitive. By a theorem of Jordan [8, Theorem 7.6A], the only sharply 4-transitive finite groups are S_4, A_6 and the Mathieu group M_{11}, acting naturally. We can eliminate S_4 since $n \ge 2m = 6$, and A_6 and M_{11} since their subgroups G_M, of order 18 and 48, are not dihedral. In case (iv) we have $n = 2m + 1 = 7$ and $|G| = 140\eta$. Now G is 3-homogeneous and hence 3-transitive, since the exceptions in Proposition 4 do not have degree 7. This is impossible, since $|G|$ is coprime to 3.

This leaves the case $m = 2$. In case (iv), putting $n = 5$ gives $J = J(5, 2)_2$, Petersen's graph. Since $|G| = 30\eta = 30$ or 60, the only possible subgroup $G \le S_5$ is $G = A_5$ with $\eta = 2$, so $\mathcal{M}$ is non-orientable. Since J has valency 3, G is an epimorphic image of the triangle group $\Delta[3, 2, p]$ where p (the face-valency) is the order of an element of A_5. Since $|\Delta[3, 2, p]| < |A_5|$ for $p < 5$, the only possibility is $p = 5$, so $\mathcal{M}$ is covered by the dodecahedron $\{5, 3\}$. Now $\Delta[3, 2, 5] \cong S_2 \times A_5$, so the only normal subgroup with quotient A_5 is the centre S_2, generated by the antipodal automorphism of $\{5, 3\}$; thus $\mathcal{M}$ is the antipodal quotient $\{5, 3\}/S_2 = \{5, 3\}_5$ of $\{5, 3\}$ [7, Section 8.6 and Table 8], as in part (d).

We may therefore assume that we are in case (i), so $J = J(n, 2)_1 = L(K_n)$ with n odd. The pairs $(M, M') \in \Gamma_1$ can be identified with the ordered triples (x, y, z) of distinct elements of N, where $M = \{x, y\}$ and $M' = \{y, z\}$; in the orientable case, G permutes these regularly and is therefore sharply 3-transitive on N. As shown by Zassenhaus [35], it follows that $n = q + 1$ for some prime power q, and G is either $PGL_2(q)$, or $M_2(q)$ with q an odd square (see Section 4); since n is odd we have $G = PGL_2(q)$, so $G_M \cong D_{q-1}$ which is non-cyclic since $q = n - 1 \ge 2m - 1 = 3$, and we obtain no orientable regular embeddings. In the non-orientable case, G is 3-transitive on N with 3-point stabilisers of order 2. Here $G_M (\cong D_{2(n-2)} \cong S_2 \times D_{n-2})$ acts naturally as $D_{2(n-2)}$ on the set $\Gamma_1(M)$ of neighbours of M, which can be identified with $M \times \overline{M}$, so it must act as S_2 on M and as D_{n-2} on $\overline{M}$. Since n is odd, the subgroup of G fixing $x, y \in M$ and $z \in \overline{M}$ therefore acts semiregularly on the remaining $n - 3$ points of $\overline{M}$, so the subgroup of G fixing any four points in N is trivial. A theorem of Gorenstein and Hughes [15] states that if a finite group G is 3-transitive but not sharply 3-transitive, with trivial 4-point stabilisers, then $G = A_6$, M_{11} or $P\Gamma L_2(2^f)$ (f prime), all acting naturally. Here $|G| = 2n(n-1)(n-2)$, so by comparing orders we see that $G = P\Gamma L_2(2^2) = S_5$ with $n = 5$, giving $J = J(5,2)_1 = L(K_5)$. Since J has valency 6, any non-orientable regular embedding has type $\{p, 6\}$ for some p, and thus has the form $\mathcal{M} = \{p, 6\}/K$ where K is the kernel of an epimorphism $\Delta = \Delta[6, 2, p] \to S_5$. Such epimorphisms correspond to triples of elements $r_i \in S_5$ (images of the generators R_i of Δ) which satisfy the relations of Δ and generate S_5. Since $R_i^2 = 1$, and S_5 is not a dihedral group, each r_i must be an involution, and thus a transposition or a double transposition. At least one r_i must be a transposition, for otherwise $\langle r_0, r_1, r_2\rangle \le A_5$. Since the degree 5 is prime, and a primitive group containing a transposition must be the symmetric group, it follows that r_0, r_1 and r_2 generate S_5 if and only if they generate a transitive group. By a rather tedious case-by-case analysis, which we omit, we then find that the only epimorphisms $\Delta \to S_5$ are given by the following three mappings of the generators R_i $(i = 0, 1, 2)$, where $a, \ldots, e$ is an arbitrary permutation of $1, \ldots, 5$:

$$R_i \mapsto r_i = \begin{cases} (ac), (ad)(bc), (de); \\ (ac)(de), (ad)(bc), (de); \\ (ab)(de), (ac), (ad)(be). \end{cases}$$

In each of these three cases, the various epimorphisms differ only by automorphisms of S_5, so they have the same kernel K and hence correspond to a single regular map $\mathcal{M}$. The element $Z = R_0R_1$ maps to a permutation $z = r_0r_1 = (abcd)$, $(abcde)$ or $(abc)(de)$ of order $p = 4$, 5 or 6 respectively, so $\mathcal{M}$ has type $\{4, 6\}$, $\{5, 6\}$ or $\{6, 6\}$. If $\mathcal{M}$ is to be an embedding of J then z must induce a p-cycle on Ω forming a closed path in J, the boundary of a z-invariant face of $\mathcal{M}$. In the first two cases, such a cycle is given by (ab, bc, cd, da) or (ab, bc, cd, de, ea), and in each case the different choices of $a, \ldots, e$ give the faces of a regular embedding of J. In the third case, however, the only 6-cycle of z on Ω is (ad, be, cd, ae, bd, ce), with consecutive vertices not adjacent, so the 1-skeleton of $\mathcal{M}$ is not J (it is, in fact, the multigraph formed by doubling the edges of Petersen's graph $J(5,2)_2$). In the first and second cases, $R_0R_1R_2$ is mapped to $(abced)$ or $(abce)$, so the Petrie polygons have length 5 or 4; since the orientation-reversing elements $(R_0R_1R_2)^5$ and $(R_0(R_1R_2)^2)^3$ lie in K, both maps are non-orientable. They have $|S_5|/12 = 10$ vertices, $|S_5|/4 = 30$ edges, and $|S_5|/2p = 15$ or 12 faces, so they have genus 7 or 10 respectively. Because of their types and Petrie lengths, they are quotients of the regular maps $\{4, 6\}_5$ and $\{5, 6\}_4$, which are Petrie duals of each other with automorphism group $S_2 \times S_5$ [7, Table 8]; being regular, with automorphism group S_5, they must be isomorphic to $\{4, 6\}_5/S_2$ and $\{5, 6\}_4/S_2$, and they form a Petrie dual pair as described in part (c) of the theorem. This deals with all cases where $G \le S_n$, and in particular it covers cases (a) and (c) of Theorem 2, where $A = S_n$.

Case (b). In Theorem 2(b), where $(n, m) = (12, 4)$, we have $A = O_{10}^-(2)$; the valency of J is 256 or 238 as $I = \{1, 3\}$ or $\{2, 4\}$, and there are no elements of these orders in $O_{10}^-(2)$ (see [5, p. 147] for its simple subgroup of index 2), so J has no regular embeddings.

By Theorem 2 we may therefore assume that $m = (n-1)/2$ with $I = m + 1 - I$, or $m = n/2$, as in cases (d) to (g) of Theorem 2; we may also assume that $G \not\le S_n$, since we dealt with subgroups $G \le S_n$ earlier, under cases (a) and (c).

Case (d). Suppose that $m = (n-1)/2$ and $I = m+1-I$, as in Theorem 2(d), so $A = S_{n+1}$. Then $m \geq 3$, for if $m = 2$ then $I = \{1, 2\}$, against our earlier assumption. Since S_{n+1} has orbitals Γ_0 and $\Gamma_i \cup \Gamma_{m+1-i}$, it follows from the transitivity of G on directed edges, and the condition $I = m+1-I$, that either $I = \{i, m+1-i\}$ for some $i \neq (m+1)/2$ or $I = \{(m+1)/2\}$. As in the proof of Theorem 2, we can identify the vertex-set Ω with the set Φ of equipartitions of $N^* = N \cup \{\infty\}$, two equipartitions being adjacent if their parts intersect in sets with cardinality in I. Since G acts transitively on Φ, either it is transitive on the $(m+1)$-element subsets of N^*, or it has two orbits on them, each consisting of the complements of the subsets in the other. If G is transitive on $(m+1)$-element subsets then by Proposition 4 it is $(m+1)$-transitive on N^*, since the exceptions to 4-transitivity do not have degree 8. The stabiliser of an $(m+1)$-element set therefore acts on it as S_{m+1}, with $m+1 \geq 4$, so the stabiliser of the corresponding equipartition is not cyclic or dihedral, a contradiction. Thus G has two orbits on $(m+1)$-element subsets. Since complementary sets are in different orbits, the stabiliser G_E of an equipartition $E = \{M, \overline{M}\}$ preserves its parts M and $\overline{M}$. If $I = \{i, m+1-i\}$ with $i \neq (m+1)/2$ then the adjacent equipartitions E' correspond to the ordered pairs of i-element subsets $I_1 \subset M$ and $I_2 \subset \overline{M}$, with $E' = \{(M \backslash I_1) \cup I_2), I_1 \cup (\overline{M} \backslash I_2)\}$, so the cyclic group G_E^* of index η in G_E must permute such pairs regularly. However, this is impossible by Lemma 6, since the actions of G_E^* on the i-element subsets of M and of $\overline{M}$ have the same degree. We may therefore assume that $I = \{i\}$ where $i = (m+1)/2$. Since $i \neq 0, 1, m$ or $m+1$, Lemma 5 implies that G_E^* has at least two orbits on the i-element subsets $I_1 \subseteq M$, and similarly for i-element subsets $I_2 \subseteq \overline{M}$, so it has at least $(2 \times 2)/2 = 2$ orbits on equipartitions $E' = \{(M \backslash I_1) \cup I_2), I_1 \cup (\overline{M} \backslash I_2)\}$ adjacent to E. This contradicts the regularity of $\mathcal{M}$, so we have covered all cases in Theorem 2 where $m < n/2$, leaving cases (e) to (g).

Now let $m = n/2$, so A acts imprimitively on Ω, permuting the set Φ consisting of the $e = d/2$ equipartitions $\{M, \overline{M}\}$ of Ω. Then G also acts imprimitively, and G_M fixes the vertices M and $\overline{M}$ of J. Since G_M acts transitively on the neighbours of M, and since we are assuming that $I \neq \{m\}$, it follows that $\overline{M}$ is not a neighbour; thus $m \notin I$, so $I' = I$.

Case (e). Suppose that $I' \neq I''$, so $A = D : S_n = S_2 \times S_n$ by Theorem 2(e). The orbitals $\Gamma_0, \ldots, \Gamma_m$ of A are invariant under G, and since G_M is transitive on the neighbours of M, it follows that $|I| = 1$, say $I = \{i\}$ where $i \neq m/2$. Since we are also assuming that $G \not\leq S_n$, the subgroup $H := G \cap S_n$ has index 2 in G. If H is transitive on directed edges then it is regular on them, and since $H \leq S_n$, our earlier arguments (applied to H rather than G) show that we are in case (i) with m and n coprime, or case (iv) with $m = (n-1)/2$, each contradicting $m = n/2$. Hence H has two orbits on directed edges, transposed by D, so either H has two orbits on vertices, with H_M transitive on the neighbours of each $M \in \Omega$, or H is transitive on vertices, with H_M having two orbits on neighbours. In the first case, since the neighbours of M are the sets $M' = (M \backslash I_1) \cup I_2$, where I_1 and I_2 are i-element subsets of M and $\overline{M}$, the cyclic group H_M^* must act transitively on ordered pairs (I_1, I_2) of such sets. This is impossible by Lemma 6, since the representations of H_M on subsets I_1 and I_2 have the same degree. We may therefore assume that H is transitive on vertices, so H acts as an m-homogeneous group on N. Since $i \neq m/2$ we have $m \geq 3$, so Proposition 4 implies that H is m-transitive on N since the exceptions to m-transitivity for $m = 3$ or 4 do not have degree $n = 6$ or 8. Thus H_M acts on M as S_m, and since H_M is cyclic or dihedral it follows that $m = 3$ and $\eta = 2$. Then $n = 6$ and $i = 1$ or 2, so $|G| = \eta\binom{6}{3}\binom{3}{1}\binom{3}{2} = 360$; thus H has order 180 and is therefore a subgroup of index 4 in S_6, which is impossible by the simplicity of A_6, so case (e) is eliminated.

Case (f). Suppose that $I' = I''$ and $I \neq \{1, \ldots, m-1\}$, so $m \geq 4$. By Theorem 2(f), $A = S_2^e : S_n$, where the direct factors of the normal subgroup $B = S_2^e$ are generated by the transpositions t_E corresponding to the equipartitions $E \in \Phi$ of N, and the complement S_n is induced by the permutations of N. The orbitals of A are $\Gamma_0, \Gamma_i \cup \Gamma_{m-i}$ ($i = 1, \ldots, \lfloor m/2 \rfloor$) and Γ_m, so the transitivity of G_M on the neighbours of M, together with the condition $I = m - I$, gives $I = \{i, m-i\}$ where $1 \leq i < m/2$ or $I = \{i\}$ where $i = m/2$. If M' is a neighbour of M then so is its complement, so G_M acts imprimitively on these neighbours, permuting complementary pairs. Under the natural epimorphism $A \to S_n$ given by the action on Φ, the image of G_M^* is a cyclic subgroup H of S_n (acting naturally on N) which preserves the equipartition $E = \{M, \overline{M}\}$ and acts transitively on the equipartitions E' adjacent to E (those with parts M' satisfying $|M \cap M'| \in m - I = I$); thus H has at most two orbits on the neighbours of M, and if there are two then each consists of the complements of the sets in the other orbit. These neighbours are the sets $M' = (M \backslash I_1) \cup I_2$, and their complements if $i < m/2$, where I_1 and I_2 are i-element subsets of M and $\overline{M}$. Now either H preserves M and $\overline{M}$, or it transposes them. First suppose that it preserves them. If $i < m/2$ then H must be transitive on ordered pairs of i-element subsets $I_1 \subset M$ and $I_2 \subset \overline{M}$, which is impossible by Lemma 6 since the actions on i-element subsets of M and $\overline{M}$ have the same degree. If $i = m/2$ then H must have at most two orbits on ordered pairs (I_1, I_2), so it is transitive on the i-element subsets of at least one of M and $\overline{M}$; since H is cyclic, Lemma 5 gives $i = 1$ and hence $m = 2$, which contradicts the fact that $m \geq 4$. The other possibility is that H transposes M and $\overline{M}$, so it acts on the set Δ of all i-element subsets $I_1 \subset M$ or $I_2 \subset \overline{M}$. It must be transitive on Δ, for if it had distinct orbits Δ_1 and Δ_2 on Δ, it would have at least three orbits on sets $M' = (M \backslash I_1) \cup I_2$, namely those with I_1, I_2 both in Δ_1, both in Δ_2, or one in each. Hence the subgroup of index 2 in H, which leaves M and $\overline{M}$ invariant, acts transitively on the i-element subsets of each of these sets, so Lemma 5 gives $i = 1$ since $1 \leq i \leq m/2 < m - 1$. Thus we can identify Δ with N, so a generator h of H permutes N in a single cycle of length $n = 2m \geq 8$. It follows that H has at least three orbits on neighbours $M' = (M \backslash I_1) \cup I_2$ of M, since sets of the form $M' = (M \backslash \{x\}) \cup \{xh^j\}$ must be in different orbits for $j = 1$, 3 and 5. This is a contradiction, so case (f) of Theorem 2 is dealt with.

Case (g). Let $I = \{1, \ldots, m-1\}$, so Theorem 2(g) gives $A = S_2^e : S_e = S_2 \wr S_e$, a group which acts imprimitively on Ω by permuting the set Φ of complementary pairs $E = \{M, \overline{M}\} \subset \Omega$. It follows that G_M fixes M and $\overline{M}$ and acts imprimitively on the remaining vertices M' of J, again permuting complementary pairs. These $d - 2$ vertices M' are the neighbours of M, and are therefore permuted regularly by G_M^*, so if g is a generator of G_M^* then $g^{(d-2)/2}$ is an involution fixing M and $\overline{M}$ and transposing all other complementary pairs. This is the element t_Ψ of the base group $B = S_2^e$ of $S_2 \wr S_e$, where $\Psi = \Phi \backslash \{E\}$; as M ranges over Ω the elements t_Ψ generate a subgroup of index 1 or 2 in B as e is even or odd, so $|G \cap B|$ is divisible by 2^{e-1}. Since G^M acts transitively on $\Omega \backslash \{E\}$,

G induces a doubly transitive group $G/(G \cap B)$ of degree e on Φ, so $e(e-1)$ divides the index $|G : G \cap B|$ and hence $|G|$ is divisible by $2^{e-1}e(e-1)$. However, J has d vertices of valency $d-2$, so $|G| = \eta d(d-2) = 4\eta e(e-1) \leq 8e(e-1)$ and hence $e \leq 4$. Since $e = \binom{2m}{m}/2$ with $m \geq 2$ we must have $m = 2$ and $e = 3$, so $J = J(4,2)_1$ with $A = S_2^3 : S_3 = D \times S_4$. In the orientable case we have $|G| = 24$, so $|A : G| = 2$, and of the three subgroups of index 2 in A, only one is transitive on directed edges and has cyclic vertex-stabilisers, namely the rotation group $G = \Delta(4,2,3) \cong S_2^2 : S_3 \cong S_4$ of the octahedral map $\{3,4\}$ on the sphere [7, Section 4.2]. Now $\mathcal{M}$ has type $\{p,4\}$ where p is the order of an element of G, so $p \leq 4$. One easily checks that the only possible epimorphisms $\Delta(4,2,p) \to G$ are the isomorphisms where $p = 3$, so $\mathcal{M} = \{3,4\}$ with full automorphism group $\Delta[4,2,3] = A = S_2 \times S_4$ as in part (a). In the non-orientable case we have $|G| = 48$, so $G = A$ and hence $p \leq 4$ or $p = 6$. It is easily seen that there is an epimorphism $\Delta[4,2,p] \to A$ with kernel $K \not\leq \Delta(4,2,p)$ only when $p = 6$. Now A has a single conjugacy class of elements of order 6, each a cyclic permutation of the consecutive vertices of a Petrie polygon of $\{3,4\}$. These must therefore form the four faces of $\mathcal{M}$, so $\mathcal{M}$ is the Petrie dual $\{6,4\}_3$ of $\{3,4\}$, a non-orientable map of genus 4, as in part (a). $\square$

The following three results are special cases of Theorem 10. Putting $m = 2$ and $i = 1$, and using Proposition 9 for $L(K_3) \cong K_3$, we have:

Corollary 11. *The only regular embeddings of $L(K_n)$ for $n \geq 3$ are:*

(a) *the dihedron $\{3,2\}$ on $\mathcal{S}_0^+$ and its Petrie dual $\{6,2\}_3 = \{6,2\}/S_2$ on $\mathcal{S}_1^-$ for $n = 3$,*
(b) *the octahedron $\{3,4\}$ on $\mathcal{S}_0^+$ and its Petrie dual $\{6,4\}_3$ on $\mathcal{S}_4^-$ for $n = 4$, and*
(c) *the Petrie dual pair $\{4,6\}_5/S_2$ on $\mathcal{S}_7^-$ and $\{5,6\}_4/S_2$ on $\mathcal{S}_{10}^-$ for $n = 5$.*

Putting $i = m = 2$, we have:

Corollary 12. *The only regular embedding of $\overline{L(K_n)}$ for $n \geq 5$ is the embedding $\{5,3\}_5 = \{5,3\}/S_2$ of Petersen's graph $\overline{L(K_5)}$ in $\mathcal{S}_1^-$.*

Similarly, putting $i = m = (n-1)/2$:

Corollary 13. *The only regular embedding of an odd graph O_{m+1} of valency $m+1 \geq 3$ is the embedding $\{5,3\}_5 = \{5,3\}/S_2$ of Petersen's graph O_3 in $\mathcal{S}_1^-$.*

7. Vertex-transitive embeddings

For most of the merged Johnson graphs $J = J(n,m)_I$, the arguments used to prove Theorem 10 actually yield a stronger result, that J has no vertex-transitive embeddings. The critical point is that for any map, regular or not, the automorphisms fixing a vertex form a cyclic or dihedral group.

Theorem 14. *Let $J = J(n,m)_I$ where $5 \leq m < (n-1)/2$ and $\emptyset \subset I \subset \{1,\ldots,m\}$. Then J has no vertex-transitive embeddings in orientable or non-orientable surfaces.*

Proof. Given such an embedding, its automorphism group G is a subgroup of $A = \mathrm{Aut}\, J$ which acts transitively on Ω. By Theorem 2(a), the conditions on m imply that $A = S_n$, so G is an m-homogenous permutation group on N. Since $m \geq 5$, Proposition 4 implies that G is m-transitive, so the stabiliser G_M of a vertex M induces S_m on the subset M and cannot therefore be cyclic or dihedral. □

This argument can be extended to the case $4 = m < (n-1)/2$, provided one avoids the values $n = 12$ and 33, where the groups $G = O_{10}^{-}(2)$ and $P\Gamma L_2(32)$ are exceptions in Theorem 2 and Proposition 4 respectively. It also applies if $5 \leq m = (n-1)/2$ and $I \neq m + 1 - I$, since Theorem 2(c) again gives $A = S_n$.

8. Constructions

Some of the maps of higher genus appearing in Theorem 10 may be unfamiliar, so here we give rather more detailed constructions for them.

There is an epimorphism $\theta : \Delta[5, 2, 4] \to \{\pm 1\} \times S_5 \cong S_2 \times S_5$ given by

$$R_0 \mapsto -(35), \qquad R_1 \mapsto -(25)(43), \qquad R_2 \mapsto -(12)(35),$$

which extends the epimorphism $\Delta(5, 2, 4) \to S_5$ given by

$$X \mapsto (12345), \qquad Y \mapsto (12), \qquad Z \mapsto (2543),$$

so as in Section 5, the kernel K of θ corresponds to an orientable reflexible map $\mathcal{M} = \{4, 5\}/K$ of type $\{4, 5\}$ with $\mathrm{Aut}\,\mathcal{M} \cong S_2 \times S_5$. Having $120/5 = 24$ vertices, $120/2 = 60$ edges and $120/4 = 30$ faces, $\mathcal{M}$ has Euler characteristic -6 and hence genus 4. Since the image $-(123)(45)$ of $R_0R_1R_2$ has order 6, $\mathcal{M}$ is a quotient of $\{4, 5\}_6$, and since $\mathrm{Aut}\{4, 5\}_6 \cong S_2 \times S_5$ [7, Table 8] these two maps are isomorphic. Now $\mathcal{M}$ and its dual $\mathcal{M}' = \{5, 4\}_6$ project onto regular maps $\mathcal{M}/S_2$ and $\mathcal{M}'/S_2$ with automorphism group S_5 on the surface $\mathcal{S}_4^+/S_2 = \mathcal{S}_5^-$, and the duals of their Petrie duals $\{6, 5\}_4/S_2$ and $\{6, 4\}_5/S_2$ are the maps $\{5, 6\}_4/S_2$ and $\{4, 6\}_5/S_2$ in Theorem 10(c). Alternatively, these maps can be obtained directly from $\mathcal{M}'/S_2$ and $\mathcal{M}/S_2$ by applying Wilson's 'opposite' operation [20, 21, 34]: this transposes vertices and Petrie polygons, while preserving edges and faces, by cutting a map along its edges and then rejoining adjacent faces with reversed orientation.

The inverse image of $S_2 \times A_5$ under θ is a triangle group $\Delta[5, 2, 5]$, the subgroup of index 2 in $\Delta[5, 2, 4]$ generated by the reflections $R_0' = R_0R_1R_0$, $R_1' = R_2$ and $R_2' = R_1$. Since this contains K there is a reflexible map $\mathcal{N} = \{5, 5\}/K$ of type $\{5, 5\}$ on $\mathcal{S}_4^+$, with automorphism group $S_2 \times A_5$. This is the great dodecahedron $\{5, 5/2\}$, isomorphic as a map to its dual, the small stellated dodecahedron $\{5/2, 5\}$ [6, Sections 6.2 and 6.6], and $\mathcal{N}/S_2$ is the map $\{5, 5\}_3$ on $\mathcal{S}_5^-$ appearing in Theorem 10(b). One can construct $\mathcal{M} = \{4, 5\}_6$ from $\mathcal{N}$ by adding edges joining each of the twelve vertices of $\mathcal{N}$ to the centres of its five incident faces, and then deleting the edges of $\mathcal{N}$; thus the maps in Theorem 10(c) can also be obtained from $\mathcal{N}$, with their extra automorphisms induced by the self-duality of $\mathcal{N}$.

Alternatively, one can construct conformal models of $\mathcal{M}$ and $\mathcal{N}$. The surface group K acts as a group of isometries of the hyperbolic plane, and the quotient space is a Riemann surface of genus 4 isomorphic to Bring's curve, the complex projective variety $\mathcal{B} \subset \mathbf{P}^4(\mathbf{C})$ given by $\sum_{j=1}^{5} z_j^k = 0$ $(k = 1, 2, 3)$. The conformal automorphisms of $\mathcal{B}$ form a group S_5, permuting the homogeneous coordinates z_j; this commutes with the anticonformal involution induced by complex conjugation, and together they generate the isometry group

Iso $\mathcal{B} = S_2 \times S_5$ of the hyperbolic 2-manifold $\mathcal{B}$. The map $\mathcal{M}$ can be drawn on $\mathcal{B}$, with Aut $\mathcal{M}$ = Iso $\mathcal{B}$, by taking the vertices, edge-centres and face-centres to be the images under S_5 of the points $[1, \zeta, \zeta^2, \zeta^3, \zeta^4]$, $[1, 1, \alpha, \beta, \gamma]$ and $[0, 1, i, -1, -i]$ fixed by X, Y and Z, where ζ and i are primitive 5th and 4th roots of unity, and α, β, γ are the roots of the polynomial $z^3 + 2z^2 + 3z + 4$. These vertices form two orbits under A_5, consisting of the vertices and face-centres of $\mathcal{N}$, while its edge-centres are the face-centres of $\mathcal{M}$. The edges of $\mathcal{M}$ and $\mathcal{N}$ are the images under S_5 and A_5 of line-segments on $\mathcal{B}$ fixed by the reflections

$$R_2 : [z_1, z_2, z_3, z_4, z_5] \mapsto [\overline{z}_2, \overline{z}_1, \overline{z}_5, \overline{z}_4, \overline{z}_3]$$

and

$$R_2' : [z_1, z_2, z_3, z_4, z_5] \mapsto [\overline{z}_1, \overline{z}_5, \overline{z}_4, \overline{z}_3, \overline{z}_2].$$

The connections between Bring's curve and the small stellated dodecahedron $\{5/2, 5\}$ are explored in detail by Weber in [31], developing earlier ideas of Kepler and Klein.

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References

[1] M. Aigner, G. Ziegler, Proofs from THE BOOK, Springer, Berlin, 1998.
[2] N.L. Biggs, Automorphisms of imbedded graphs, J. Combin. Theory Ser. B 11 (1971) 132–138.
[3] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.
[4] M. Brown, Weak n-homogeneity implies weak $n-1$-homogeneity, Proc. Amer. Math. Soc. 10 (1959) 644–647.
[5] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, ATLAS of Finite Groups, Clarendon Press, Oxford, 1985.
[6] H.S.M. Coxeter, Regular Polytopes, Dover, New York, 1973.
[7] H.S.M. Coxeter, W.O.J. Moser, Generators and Relations for Finite Groups, 3rd edition, Springer, Berlin, 1972.
[8] J.D. Dixon, B. Mortimer, Permutation Groups, Springer-Verlag, New York, 1996.
[9] R. Dye, Symmetric groups as maximal subgroups of orthogonal and symplectic groups over the field of two elements, J. London Math. Soc. 20 (1979) 227–237.
[10] H. Enomoto, Characterization of families of finite permutation groups by the subdegrees, I, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 19 (1972) 129–135.
[11] P. Erdős, On a diophantine equation, J. London Math. Soc. 26 (1951) 176–178.
[12] I.A. Faradžev, M.H. Klin, M.E. Muzichuk, Cellular rings and groups of automorphisms of graphs, in: I.A. Faradzev, A.A. Ivanov, M.H. Klin, A.J. Woldar (Eds.), Investigations in Algebraic Theory of Combinatorial Objects, Kluwer, Dordrecht, 1994, pp. 1–152.
[13] J.S. Frame, The double cosets of a finite group, Bull. Amer. Math. Soc. 47 (1941) 458–467.
[14] G. Glauberman, Normalizers of p-subgroups in finite groups, Pacific J. Math. 29 (1969) 137–144.
[15] D. Gorenstein, D.R. Hughes, Triply transitive groups in which only the identity fixes four letters, Illinois J. Math. 5 (1961) 486–491.
[16] E. Halberstadt, On certain maximal subgroups of symmetric or alternating groups, Math. Z. 151 (1976) 117–125.

[17] A.A. Ivanov, Distance-transitive graphs and their classification, in: I.A. Faradžev, A.A. Ivanov, M.H. Klin, A.J. Woldar (Eds.), Investigations in Algebraic Theory of Combinatorial Objects, Kluwer, Dordrecht, 1994, pp. 283–378.
[18] L.D. James, Imbeddings of the complete graph, Ars Combin. 16-B (1983) 57–72.
[19] L.D. James, G.A. Jones, Regular orientable imbeddings of complete graphs, J. Combin. Theory Ser. B 39 (1985) 353–367.
[20] G.A. Jones, D. Singerman, Maps, hypermaps and triangle groups, in: L. Schneps (Ed.), The Grothendieck Theory of Dessins d'Enfants, London Math. Soc. Lecture Note Ser., vol. 200, 1994, pp. 115–145.
[21] G.A. Jones, J.S. Thornton, Operations on maps, and outer automorphisms, J. Combin. Theory Ser. B 35 (1983) 93–103.
[22] L.A. Kalužnin, M.H. Klin, On some maximal subgroups of the symmetric and alternating groups, Mat. Sbornik 87 (1972) 91–121 (Russian); Math. USSR-Sb. 16 (1972) 95–124 (English transl.).
[23] W. Kantor, k-homogeneous groups, Math. Z. 124 (1972) 261–265.
[24] M.H. Klin, Investigations of the algebras of invariant relations of some classes of permutation groups, Ph.D. Thesis, Nikolaev, 1974 (in Russian).
[25] D. Livingstone, A. Wagner, Transitivity of finite permutation groups on unordered sets, Math. Z. 90 (1965) 393–403.
[26] R. Nedela, M. Škoviera, Regular embeddings of canonical double coverings of graphs, J. Combin. Theory Ser. B 67 (1996) 249–277.
[27] C.C. Sims, Graphs and finite permutation groups, I, Math. Z. 95 (1967) 76–86.
[28] J. Skalba, On the maximality of S_n in $S_{\binom{n}{k}}$, J. Algebra 75 (1982) 158–174.
[29] V.A. Ustimenko-Bakumovskiĭ, The lattice of supergroups of an induced symmetric group, Dokl. Akad. Nauk SSSR 237 (1977) 276–278 (in Russian); Soviet Math. Dokl. 18 (1977) 1433–1437 (English transl.).
[30] V.A. Ustimenko-Bakumovskiĭ, Induced maximal permutation groups, Ph.D. Thesis, Kiev, 1979 (in Russian).
[31] M. Weber, On Kepler's small stellated dodecahedron, Pacific J. Math. (in press).
[32] H. Whitney, Congruent graphs and the connectivity of graphs, Amer. J. Math. 54 (1932) 150–168.
[33] H. Wielandt, Finite Permutation Groups, Academic Press, New York, 1964.
[34] S.E. Wilson, Operators over regular maps, Pacific J. Math. 81 (1979) 559–568.
[35] H. Zassenhaus, Kennzeichnung endlicher linearer Gruppen als Permutationsgruppen, Abh. Math. Sem. Hamburg 11 (1936) 17–40.